Fall 2009 Math 245 Exam 3 Solutions

Exam scores: One quarter of the exam scores were below 60, one quarter between 60 and 65 (the median), one quarter between 65 and 70, and one quarter of the scores were above 70. This is a significant dip as compared to the first two exams, and was predicted by a similar dip in quiz scores. Students are urged to take incorrect quizzes as a call to action, whether or not they were chosen to hand these quizzes in.

1. Carefully define the following terms:
   This problem tests the students' attention to detail and commitment to accurate definitions, which are very important in mathematics. A function is a subset of $A \times B$; the second set B is the *codomain*. An *injection* is a function f that satisfies: for all x, y in the domain, if $x \neq y$ then $f(x) \neq f(y)$. A *partially ordered set* is a set with a relation that is reflexive, antisymmetric, and transitive. The *l.u.b.*, or least upper bound, of a set is the (unique) minimal element among all the upper bounds of that set. NOTE: the book has a typo in this definition; as written it makes no sense. The *strong pigeonhole principle* states that if n pigeons are distributed into m pigeonholes, then at least one hole receives at least $\lceil \frac{n}{m} \rceil$ pigeons.

2. Prove or disprove the following statement. For all functions $f : \mathbb{N} \to \mathbb{N}$, if f is injective then f is surjective.
   This problem tests understanding of the limitations of the injective-surjective theorem on finite sets. Its solution is similar to many homework exercises, such as Problem 21.1. The statement is false; a counterexample is $f(n) = n + 1$. This is injective [if $f(n) = f(m)$, then $n + 1 = m + 1$, so $n = m$], but not surjective [no element is mapped to 1].

3. Find all equivalence relations on $A = \{x, y\}$.
   This problem tests the definition of equivalence relations. All equivalence relations on A are reflexive, hence they must contain (x, x) and (y, y). Because they are symmetric, if they contain (x, y) then they must contain (y, x) and vice versa. Hence there are two equivalence relations: $R = \{(x, x), (y, y)\},\ S = \{(x, x), (y, y), (x, y), (y, x)\}$.

4. Find all posets on $A = \{x, y\}$.
   This problem tests the definition of posets. Like in problem 3, all posets (being reflexive) must contain (x, x) and (y, y). Because posets are antisymmetric, if they contain (x, y) then they must NOT contain (y, x) and vice versa. Hence there are three posets: $R = \{(x, x), (y, y)\},\ T = \{(x, x), (y, y), (x, y)\},\ U = \{(x, x), (y, y), (y, x)\}$.

5. Find all functions $f : A \to A$, for $A = \{x, y\}$.
   This problem tests the definition of functions. Each function must take on exactly one value at x, and exactly one value at y. There are two possibilities for each of these choices, hence there are $2 \times 2 = 4$ functions: $f_1 = \{(x, x), (y, x)\},\ f_2 = \{(x, x), (y, y)\},\ f_3 = \{(x, y), (y, x)\},\ f_4 = \{(x, y), (y, y)\}$.

6. How many relations are there on $A = \{x, y\}$? Give two examples, neither of which are equivalence relations, posets, or functions.
   This problem tests the definition of relation. A relation is a subset of $A \times A$ (equivalently, an element of the power set of $A \times A$). $|A \times A| = 4$, so there are $2^4 = 16$ possible subsets. The previous problems found seven of these subsets (one appeared three times), so there are nine others

to choose from. $R_1 = \emptyset,\ R_2 = \{(x,x)\},\ R_3 = \{(x,y)\},\ R_4 = \{(y,x)\},\ R_5 = \{(y,y)\},\ R_6 = \{(x,x),(x,y)\},\ R_7 = \{(y,x),(y,y)\},\ R_8 = \{(x,x),(x,y),(y,x)\},\ R_9 = \{(y,y),(x,y),(y,x)\}$.

7. Solve the recurrence given by $a_0 = 3, a_1 = -3, a_n = -4a_{n-1} - 4a_{n-2} (n \geq 2)$.
   This problem tests solution of second-order linear recurrence relations with constant coefficients. This one has characteristic equation $t^2 = -4t - 4$, which rearranges to $(t+2)^2 = 0$. Hence $t = -2$ is a double root and the general solution is $a_n = A(-2)^n + Bn(-2)^n$. Using the initial conditions, we get $3 = a_0 = A(-2)^0 + B0(-2)^0 = A$, and $-3 = a_1 = A(-2)^1 + B1(-2)^1 = -2A - 2B$. We plug in $A = 3$ to find $B = -1.5$. Hence the solution is $a_n = 3(-2)^n - 1.5n(-2)^n$.

8. Prove that $f : \mathbb{N} \to \mathbb{Z}$ is injective, where $f(n) = \left\{\begin{matrix} n/2 & n \text{ even} \\ (1-n)/2 & n \text{ odd} \end{matrix}\right\}$.
   This problem tests proofs of injectivity. Suppose that $f(n) = f(m)$. To prove injectivity, we need to prove that $n = m$. There are three cases. (1) n, m are both even. Then $f(n) = n/2, f(m) = m/2$, so $n/2 = m/2$ and $n = m$. (2) n, m are both odd. Then $f(n) = (1-n)/2, f(m) = (1-m)/2$, so $(1-n)/2 = (1-m)/2$ and again $n = m$. (3) m, n are of opposite parity; without loss of generality assume n is even and m is odd. Then $f(n) = n/2, f(m) = (1-m)/2$, so $n/2 = (1-m)/2$, $n = 1-m$, and $m+n = 1$. But $m, n \in \mathbb{N}$, so it is not possible for their sum to be 1, so this case never happens.

9. We define a lattice on $A = \mathbb{R} \times \mathbb{R}$ as follows. For $x = (x_1, x_2), y = (y_1, y_2)$, elements of A, we say $x \leq y$ if $\big(x_1 \leq y_1 \text{ AND } x_2 \leq y_2\big)$. For $x = (0.5, 3), y = (4, 1)$, find l.u.b.(x, y) and g.l.b.(x, y).
   This problem tests the definition of lattices, but mostly it tests courage in the face of a complicated definition. Let $w = (w_1, w_2)$ be the lub. Because $w \geq x$, we have $w_1 \geq x_1 = 0.5, w_2 \geq x_2 = 3$. Because $w \geq y$, we have $w_1 \geq y_1 = 4, w_2 \geq y_2 = 1$. Simplifying these four inequalities we get $w_1 \geq 4, w_2 \geq 3$. There are many w that would work, but there is a unique minimal w, namely $(4, 3)$. Similarly, $(0.5, 1)$ is the glb.

10. Find a finite-state automaton on $\Sigma = \{a, b\}$ that recognizes those words with exactly one b (and no other words).
    This problem tests understanding of finite-state automata. Many solutions are possible, but the simplest deterministic one is shown below. Correct non-deterministic machines also received full credit. Some students interpreted the problem to mean an automaton that accepts the word 'b' and no other words at all; although this was not my intent this is a reasonable interpretation and these solutions also received full credit.